POSTECH 이성익 교수의
양자 세계에 관한 강연
- 19장 -

편집 도우미: **POSTECH** 학부생 임향택

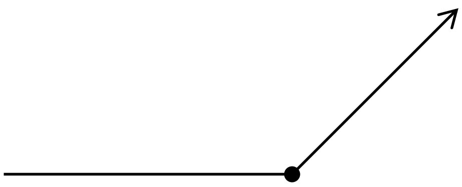
Chapter 19

Collision Theory

정리 : $e^{i\vec{k}\cdot\vec{r}} = \frac{i}{2k}\sum_{l=0}^{\infty}(2l+1)\cdot i^{l}\cdot\left[\frac{e^{-i\left(kr-\frac{l\pi}{2}\right)}}{r}-\frac{e^{i\left(kr-\frac{l\pi}{2}\right)}}{r}\right]P_{l}(\cos\theta)$

Interactions of elementary particles

Scattering theory



atom의 lifetime은 매우 길다고 가정

Collision Cross Section

Differential cross section

$$= \frac{\text{\# of particle}}{\text{Solid angle} \cdot \text{time}}$$

$$\vec{j} = \frac{\hbar}{2im}\left(\psi^{*}\nabla\psi - \psi\nabla\psi^{*}\right)$$

$$= \frac{\hbar k}{m}$$

$$e^{i\vec{k}\cdot\vec{r}} = \frac{i}{2k}\sum_{l=0}^{\infty}(2l+1)\cdot i^{l}\cdot\left[\frac{e^{-i\left(kr-\frac{l\pi}{2}\right)}}{r}-\frac{e^{-i\left(kr-\frac{l\pi}{2}\right)}}{r}\right]P_{l}(\cos\theta)$$

Radial potential

$$\psi(\vec{r}) = \frac{i}{2k}\sum_{l=0}^{\infty}(2l+1)\cdot i^{l}\cdot\left[\frac{e^{-i\left(kr-\frac{l\pi}{2}\right)}}{r} - S_l(k)\cdot\frac{e^{i\left(kr-\frac{l\pi}{2}\right)}}{r}\right]P_l(\cos\theta) \qquad |S_l(k)| = 1$$

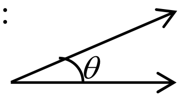
$$= \frac{i}{2k}\sum_{l=0}^{\infty}(2l+1)\cdot i^{l}\cdot\left[\frac{e^{-i\left(kr-\frac{l\pi}{2}\right)}}{r} - \frac{e^{i\left(kr-\frac{l\pi}{2}\right)}}{r} + \frac{e^{i\left(kr-\frac{l\pi}{2}\right)}}{r} S_l(k)\cdot\frac{e^{i\left(kr-\frac{l\pi}{2}\right)}}{r}\right]P_l(\cos\theta)$$

$$= e^{i\vec{k}\cdot\vec{r}} + \left[\sum_{l=0}^{\infty}(2l+1)\cdot i^{l}\cdot\frac{i}{2k}\left(1-S_l(k)\right)\cdot\frac{e^{i\left(kr-\frac{l\pi}{2}\right)}}{r}\right]P_l(\cos\theta)$$

$$= e^{i\vec{k}\cdot\vec{r}} + \left[\sum_{l=0}^{\infty}(2l+1)\cdot\frac{S_l(k)-1}{2ik}P_l(\cos\theta)\right]\cdot\frac{e^{ikr}}{r}$$

m: reduced mass $\qquad e^{-i\cdot\frac{l\pi}{2}} = (-i)^{l}$

θ:



asymptotic form of a plane wave + an incoming spherical wave

Current를 계산하자.

$$\vec{j} = \frac{\hbar}{2im}\left\{\left[e^{i\vec{k}\cdot\vec{r}} + f(\theta)\cdot\frac{e^{ikr}}{r}\right]^{*}\nabla\left[e^{i\vec{k}\cdot\vec{r}} + f(\theta)\cdot\frac{e^{ikr}}{r}\right] - c.c.\right\}$$

$$= \frac{\hbar}{2im}\left\{\left[e^{-i\vec{k}\cdot\vec{r}} + f^{*}(\theta)\cdot\frac{e^{-ikr}}{r}\right]\cdot\left[i\vec{k}e^{i\vec{k}\cdot\vec{r}} + \hat{\theta}\cdot\frac{1}{r}\frac{\partial f(\theta)}{\partial\theta}\frac{e^{ikr}}{r} + \hat{r}\cdot f(\theta)\left(ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^{2}}\right)\right] - c.c.\right\}$$

$$= \frac{\hbar}{2im}\left[i\vec{k} + i\vec{k}f^{*}(\theta)\cdot\frac{e^{-ikr(1-\cos\theta)}}{r} + ik\hat{r}f(\theta)\frac{e^{ikr(1-\cos\theta)}}{r} + ik\hat{r}\left|f(\theta)\right|^{2}\cdot\frac{1}{r^{2}} - \hat{r}f(\theta)\frac{e^{ikr(1-\cos\theta)}}{r}\right.$$

$$\left. + \hat{\theta}\frac{\partial f(\theta)}{\partial\theta}\frac{e^{ikr(1-\cos\theta)}}{r^{2}} - c.c.\right]$$

$$
\begin{aligned}
\vec{j} &= \frac{\hbar\vec{k}}{m} + \frac{\hbar k}{m}\hat{r}\left|f(\theta)\right|^2 \cdot \frac{1}{r^2} \\
&+ \frac{\hbar\vec{k}}{2m}\cdot\frac{1}{r}\left[f^*(\theta)\cdot e^{-ikr(1-\cos\theta)} + f(\theta)\cdot e^{ikr(1-\cos\theta)}\right] \\
&+ \frac{\hbar k}{2m}\cdot\frac{\hat{r}}{r}\left[f^*(\theta)\cdot e^{-ikr(1-\cos\theta)} + f(\theta)\cdot e^{ikr(1-\cos\theta)}\right] \\
&- \frac{\hbar}{2im}\cdot\frac{\hat{r}}{r^2}\left[f(\theta)\cdot e^{ikr(1-\cos\theta)} - f^*(\theta)\cdot e^{-ikr(1-\cos\theta)}\right] \\
&+ \frac{\hbar}{2im}\cdot\frac{\hat{\theta}}{r^2}\left[\frac{\partial f(\theta)}{\partial\theta}e^{ikr(1-\cos\theta)} - \frac{\partial f^*(\theta)}{\partial\theta}e^{-ikr(1-\cos\theta)}\right]
\end{aligned}
$$

$\theta \neq 0$일 때

flux over a finite solid angle

$$\int \sin\theta d\theta d\phi g(\theta,\phi) e^{ikr(1-\cos\theta)} \rightarrow \text{ rapidly varying}$$

$\therefore r \rightarrow \infty$이면

$$\vec{j} = \frac{\hbar\vec{k}}{m} + \frac{\hbar k}{m}\hat{r}\left|f(\theta)\right|^2 \cdot \frac{1}{r^2}$$

Radial direction을 보도록 하자.

$$\vec{j}\cdot\hat{r} = \frac{\hbar k}{m}\cdot\frac{\left|f(\theta)\right|^2}{r^2}$$

$$\vec{j}\cdot\hat{r}dA = \frac{\hbar k}{m}\frac{\left|f(\theta)\right|^2}{\cancel{r}^2}\cancel{r}^2 d\Omega$$

differential cross section $\frac{\hbar k}{m}$

$$d\sigma = \left|f(\theta)\right|^2 d\Omega$$

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2$$

Total cross section

$$\sigma_{\text{tot}}(k)=\int d\Omega\frac{d\sigma}{d\Omega}$$

$$f(\theta)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta)\qquad \text{where } f_l(k)=\frac{\left[S_l(k)-1\right]}{2ik}$$

$$=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\cdot\frac{S_l(k)-1}{2ik}P_l(\cos\theta)$$

$$\rightarrow \text{Let } S_l(k)=e^{2i\delta_l(k)}$$

$$=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\cdot\frac{e^{2i\delta_l(k)}-1}{2ik}P_l(\cos\theta)$$

$$=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\cdot\frac{e^{i\delta_l(k)}\cdot\left(e^{i\delta_l(k)}-e^{-i\delta_l(k)}\right)}{2ik}P_l(\cos\theta)$$

$$=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l(k)}\cdot\sin\delta_l(k)\cdot P_l(\cos\theta)$$

$$\sigma_{tot}=\int d\Omega\left[\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l(k)}\sin\delta_l(k)P_l(\cos\theta)\right]$$

$$\cdot\left[\frac{1}{k}\sum_{l'=0}^{\infty}(2l'+1)e^{-i\delta_l(k)}\sin\delta_{l'}(k)P_{l'}(\cos\theta)\right]$$

$$\left(\int d\Omega P_l(\cos\theta)P_{l'}(\cos\theta)=\frac{4\pi}{2l+1}\delta_{ll'}\right)$$

$$=\frac{1}{k^2}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}(2l+1)^2\cdot\sin^2\delta_l(k)$$

$$=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k)$$

$$\operatorname{Im} f(0)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\operatorname{Im}\left[e^{i\delta_l(k)}\cdot\sin\delta_l(k)\right]P_l(1)$$

$$=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k)P_l(1)\qquad\left(P_l(1)=1\right)$$

$$\operatorname{Im} f(0)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k)$$

$$=\frac{k}{4\pi}\sigma_{\text{tot}}$$

$\left|S_l(k)\right|=1$: Conservation of flux

absorption of the incident particle이 흡수, excite ...
줄어든다.

$$S_l(k)=\eta_l(k)e^{2i\delta_l(k)}$$

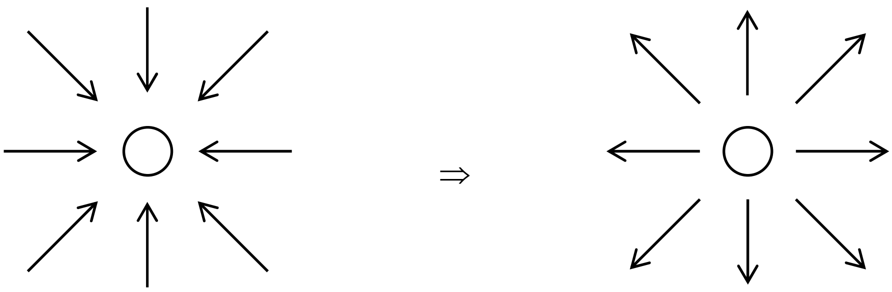
$0\le\eta_l(k)\le1$ 이 경우 흡수까지 생각할 수 있다.

$$\begin{aligned}
f_l(k)&=\frac{S_l(k)-1}{2ik}\\
&=\frac{\eta_l(k)e^{2i\delta_l(k)}-1}{2ik}\\
&=\frac{\eta_l(k)\cdot\left[\cos 2\delta_l+i\sin 2\delta_l\right]-1}{2ik}\\
&=\frac{\eta_l\cdot\sin 2\delta_l}{2k}+i\cdot\frac{1-\eta_l\cos 2\delta_l}{2k}
\end{aligned}$$

Total elastic cross section

$$\begin{aligned}
\sigma_{\text{el}}&=4\pi\sum_{l=0}^{\infty}(2l+1)\cdot\frac{\eta_l^2\sin^2 2\delta_l+\eta^2\cos^2 2\delta_l-2\eta_l\cos 2\delta_l+1}{4k^2}\\
&=4\pi\sum_{l=0}^{\infty}(2l+1)\cdot\frac{\eta_l^2-2\eta_l\cos 2\delta_l+1}{4k^2}
\end{aligned}$$

Inelastic cross section



흡수된 후 다시 나타난다.

$$\frac{i}{2k}\frac{e^{-ikr}}{r}P_l(\cos\theta) \qquad -\frac{i}{2k}S_l(k)\frac{e^{ikr}}{r}P_l(\cos\theta)$$

inward flux

$$\left(\frac{\hbar k}{m}\right)\cdot\frac{4\pi}{(2k)^2}$$

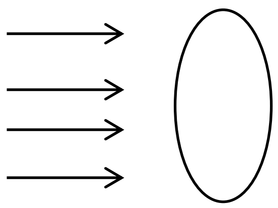
outward flux

$$\left(\frac{\hbar k}{m}\right)\frac{4\pi}{(2k)^2}\left|S_l(k)\right|^2=\left(\frac{\hbar k}{m}\right)\frac{4\pi}{(2k)^2}\left|\eta_l(k)\right|^2$$

$$\sigma_{\text{inel}} = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\underbrace{\left[1-\left|\eta_l(k)\right|^2\right]}_{\text{loss}}$$

$$\begin{aligned}\sigma_{\text{tot}} &= \sigma_{\text{el}} + \sigma_{\text{inel}} \\ &= \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\left(1+\eta_l^2 - 2\eta_l\cos 2\delta_l + 1 - \eta_l^2\right) \\ &= \frac{2\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\left(1-\eta_l\cos 2\delta_l\right)\end{aligned}$$

$$\begin{aligned}\operatorname{Im} f(0) &= \sum_{l=0}^{\infty}(2l+1)\operatorname{Im} f_l(k) \\ &= \sum_{l=0}^{\infty}(2l+1)\frac{1-\eta_l\cos\delta_l}{2k} \\ &= \frac{k}{4\pi}\sigma_{\text{tot}}\end{aligned}$$



$L = ka$

$$\begin{aligned}\sigma_{inel} &= \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\left[1-\eta_l^2(k)\right] \\ &= \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\cdot 1 \qquad \left(\because \eta_l(k)=0\right) \\ &= \frac{\pi}{k^2}\cdot\sum_{l=0}^{L}(2l+1) \\ &= \frac{\pi}{k^2}\cdot\left[\cancel{2}\cdot\frac{L(L+1)}{\cancel{2}} + (L+1)\right] \\ &= \frac{\pi}{k^2}(L+1)^2 \\ &= \frac{\pi}{k^2}k^2a^2 \\ &= \pi a^2\end{aligned}$$

$$\begin{aligned}\sigma_{\text{el}} &= 4\pi\sum_{l=0}^{L}(2l+1)\cdot\frac{1}{4k^2} \\ &= \frac{\pi}{k^2}\sum_{l=0}^{L}(2l+1) \\ &= \pi a^2\end{aligned}$$

Born Approximation

Potential이 작을 때
Energy가 클 때 excellent

Incident wave $\psi_i(\vec{r}) = \frac{1}{\sqrt{V}} e^{i\frac{\vec{p}_i \cdot \vec{r}}{\hbar}}$

Final wave $\psi_f(\vec{r}) = \frac{1}{\sqrt{V}} e^{-i\frac{\vec{p}_f \cdot \vec{r}}{\hbar}}$

p_i, p_f : initial & final momenta

Fermi의 Golden Rule

$$R_{i\to f} = \frac{2\pi}{\hbar}\int \frac{Vd^3p_f}{(2\pi\hbar)^3}|M_{fi}|^2 \cdot \underbrace{\delta\left(\frac{p_f^2}{2m}-\frac{p_i^2}{2m}\right)}_{\text{energy conservation}}$$

$$\begin{aligned}
M_{fi} &= \langle \psi_f | V | \psi_i \rangle \\
&= \int d^3r \frac{e^{-i\frac{\vec{p}_f\cdot\vec{r}}{\hbar}}}{\sqrt{V}} V(\vec{r}) \frac{e^{i\frac{\vec{p}_i\cdot\vec{r}}{\hbar}}}{\sqrt{V}} \\
&= \frac{1}{V}\int d^3r e^{-i\Delta\cdot\vec{r}} V(\vec{r}) \\
&\left(\Delta = \frac{1}{\hbar}(\vec{p}_f - \vec{p}_i)\right) \\
&= \frac{1}{V}\tilde{V}(\Delta)
\end{aligned}$$

$$\begin{aligned}
R_{i\to f} &= \frac{2\pi}{\hbar}\int d\Omega \frac{Vp_f^2dp_f}{(2\pi\hbar)^3}\frac{1}{V^2}|\tilde{V}(\Delta)|^2\delta\left(\frac{p_f^2}{2m}-E\right) \\
&= \frac{2\pi}{\hbar}\cdot\frac{1}{(2\pi\hbar)^3}\cdot\frac{1}{V}\int d\Omega p_f \cdot m \cdot \frac{p_f dp_f}{m}\delta\left(\frac{p_f^2}{2m}-E\right)|\tilde{V}(\Delta)|^2 \\
&= \frac{2\pi}{\hbar}\cdot\frac{1}{(2\pi\hbar)^3}\cdot\frac{1}{V}\int d\Omega m\cdot p_f \cdot dx\delta(x-E)|\tilde{V}(\Delta)|^2 \\
&\left(\frac{p_f^2}{2m} = x, \frac{p_f}{m}dp_f = dx\right), \ (x = E \ \therefore \text{final E should be same as initial E}) \\
&= \frac{2\pi}{\hbar}\cdot\frac{1}{(2\pi\hbar)^3}\cdot\frac{1}{V}\cdot\int d\Omega p_f \cdot m\delta(x-E)|\tilde{V}(\Delta)|^2
\end{aligned}$$

$$d\sigma = \frac{1}{4\pi^2\hbar^4}\frac{1}{|v_{\text{rel}}|} d\Omega \cdot p_f \cdot m\left|\tilde{V}(\Delta)\right|^2$$

$$|v_{\text{rel}}| = \frac{p_i}{m_1} + \frac{p_i}{m_2} = p_i\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = \frac{p_i}{m^{(i)}_{\text{reduced}}}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\cdot\frac{p_f}{p_i} m^{(f)}_{\text{red}} \cdot m^{(i)}_{\text{red}} \cdot \left|\frac{1}{\hbar^2}\tilde{V}(\Delta)\right|^2$$

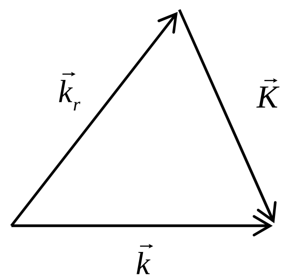
the initial and final particles are same

$$\frac{d\sigma}{d\Omega} = \frac{m^2_{\text{reduced}}}{4\pi^2}\left|\frac{1}{\hbar^2}\tilde{V}(\Delta)\right|^2$$

Rutherford Scattering

$$V = \frac{ZZ'e^2}{r}$$

Let $V = \lim_{\alpha\to 0}\beta\cdot\frac{e^{-\alpha r}}{r}$



$$\begin{aligned}
f_{1\text{B}}\left(\vec{k},\vec{k}_r\right) &= -\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\int e^{i\vec{K}\cdot\vec{r}'}V\left(\vec{r}'\right)d^3r' \\
&= -\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\cdot 2\pi\int_0^\infty r'^2dr'V\left(r'\right)\underbrace{\int_{-1}^{1}d^{iKr'\mu}d\mu}_{\frac{e^{iKr'}-e^{-iKr'}}{iKr'}} \\
&= -\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\cdot 2\pi\int_0^\infty \cancel{r'}^2dr'\cdot\frac{e^{iKr'}-e^{-iKr'}}{iK\cancel{r'}}\cdot\frac{ZZ'e^2}{\cancel{r'}}e^{-\alpha r'} \\
&= -\frac{\mu}{\hbar^2}ZZ'e^2\cdot\frac{1}{iK}\int_0^\infty\left(e^{iKr}-e^{-iKr}\right)e^{-\alpha r}dr \\
&= -\frac{\mu}{\hbar^2}ZZ'e^2\cdot\frac{1}{iK}\left[\frac{e^{(iK-\alpha)r}}{iK-\alpha}+\frac{e^{-(iK+\alpha)r}}{iK+\alpha}\right]_0^\infty \\
&= -\frac{\mu}{\hbar^2}ZZ'e^2\cdot\frac{1}{iK}\left[\frac{-1}{iK-\alpha}+\frac{-1}{iK+\alpha}\right] \\
&= -\frac{\mu}{\hbar^2}ZZ'e^2\cdot\frac{1}{iK}\frac{-iK-\cancel{\alpha}-iK+\cancel{\alpha}}{-K^2-\alpha^2} \\
&= -\frac{\mu}{\hbar^2}ZZ'e^2\cdot 2\cdot\frac{1}{K^2+\alpha^2}
\end{aligned}$$

Since $\alpha = 0$

$$f(\theta) = -\frac{2\mu ZZ' e^2}{\hbar^2 K^2}$$

$$\xrightarrow[k]{} -\frac{2\mu ZZ' e^2}{\hbar^2 \cdot 4k^2 \sin^2 \frac{\theta}{2}} \qquad \left(\because K = 2k \sin \frac{\theta}{2} \right)$$

For Coulomb Potential

Classical model

Exact Solution — all give exactly same result

1^{st} Born Approximation